But $\Delta_n - \Delta_0 = \Delta_0^2 + \Delta_1^2 + \ldots + \Delta_{n-1}^2 = \Delta_0^2(1+q+q^2+\ldots+q^{n-1})$

$$= \Delta_0^2 \frac{q^n-1}{q-1} = \frac{\Delta_0^2}{(q-1)^2} \cdot (q^n-1)(q-1),$$

and $\Delta_n^2 - \Delta_0^2 = \Delta_0^2(q^n-1)$.

Hence, substituting in [2],

$$_n\Delta_0^2 = \frac{\Delta_0^2}{(q-1)^2}\{n.(q^n-1)(q-1)+(q^n-1)^2-n.(q^n-1)(q-1)\}$$

$$= \frac{\Delta_0^2}{(q-1)^2}(q^n-1)^2.$$

$$\therefore \quad \frac{_n\Delta_0^2}{(q^n-1)^2} = \frac{\Delta_0^2}{(q-1)^2} = \frac{_n\Delta_0 - n.\Delta_0}{(q^n-1)-n.(q-1)}.$$

By means of these equations the values of Δ_0 and Δ_0^2 may be determined from the values of $_n\Delta_0$ and $_n\Delta_0^2$. The successive terms of Δ_x^2 may then be calculated from the equation $\Delta_x^2 = \Delta_0^2 . q^x$.

On the Value of Apportionable Annuities; or of Annuities in which a proportionate part is payable up to the day of death. By THOMAS BOND SPRAGUE, M.A., *Actuary of the Equity and Law Life Assurance Society, and Vice-President of the Institute of Actuaries.*

IN former papers, I have investigated the value of an annuity on a single life, when payable half-yearly, quarterly, &c. It still remains to consider the values of annuities on the joint duration of two or more lives, or on the life of the last survivor of several lives, under similar circumstances. It will, however, be more useful to consider first the increase in the value of an annuity on a single life, when a proportionate part of the annuity is payable up to the day of death, instead of the annuity ceasing, as is generally assumed, with the payment that precedes the date of death.

The value of this increase will first be investigated on the supposition that the deaths in each year of age are uniformly distributed over the year.

Thus, the annuity being £1, payable yearly, and the age of the life on which the annuity depends being k, if death occur during the $(t+1)$th year—in the instant dx following the time $t+x$, then there becomes payable the sum £x. The deaths being uniformly distributed, the chance of this sum becoming payable, is

$$\frac{l_{k+t} - l_{k+t+1}}{l_k} \cdot dx = (p_{k,t} - p_{k,t+1})dx,$$

and the value of the chance of receiving it is therefore

$$xv^{x+t}(p_{k,t}-p_{k,t+1})dx.$$

The value of all such possible payments in the $(t+1)$th year will be found by integrating, and will be equal to

$$\int_0^1 xv^{x+t}(p_{k,t}-p_{k,t+1})dx$$
$$=v^t(p_{k,t}-p_{k,t+1})\int_0^1 xv^x dx.$$

Again, the value of all such possible payments throughout life will be found by summing this quantity with regard to t, giving t the values 0, 1, 2 . . . to the extremity of life.

Now $$\Sigma_t v^{t+1}(p_{k,t}-p_{k,t+1})=\mathrm{A}_k.$$

Hence the value of the increase, which it will be convenient to call the "correction," becomes

$$\frac{\mathrm{A}_k}{v}\int_0^1 xv^x dx, \text{ or } \mathrm{A}_k\int_0^1 xv^{x-1}dx.$$

The value of the definite integral involved may be found as follows:—

$$xv^{x-1}=x(1+i)^{-\overline{x-1}}$$
$$=x-x(x-1)i+x(x-1)x\frac{i^2}{2}-x(x-1)x(x+1)\frac{i^3}{6}+\ldots.$$

(by the Binomial Theorem)

$$=x-(x^2-x)i+(x^3-x^2)\frac{i^2}{2}-(x^4-x^2)\frac{i^3}{6}+\ldots.$$

whence

$$\int xv^{x-1}dx=\mathrm{C}+\frac{x^2}{2}-\left(\frac{x^3}{3}-\frac{x^2}{2}\right)i+\left(\frac{x^4}{4}-\frac{x^3}{3}\right)\frac{i^2}{2}-\left(\frac{x^5}{5}-\frac{x^3}{3}\right)\frac{i^3}{6}+\ldots(1)$$

and

$$\int_0^1 xv^{x-1}dx=\frac{1}{2}+\frac{i}{6}-\frac{i^2}{24}+\frac{i^3}{45}-\quad\ldots\ldots\ldots\quad(2)$$

Or, otherwise, integrating by parts,

$$\int xv^x dx=\frac{xv^x}{\log v}-\int\frac{v^x}{\log v}dx$$
$$=\frac{xv^x}{\log v}-\frac{v^x}{\log^2 v}+\mathrm{C}$$
$$=\mathrm{C}-\frac{xv^x}{\delta}-\frac{v^x}{\delta^2}\quad\ldots\ldots\ldots\quad(2^*)$$
$$\int_0^1 xv^x dx=\frac{1}{\delta^2}-\frac{v}{\delta}-\frac{v}{\delta^2},$$

and $$\int_0^1 xv^{x-1}dx = \frac{1}{v\delta^2} - \frac{1}{\delta} - \frac{1}{\delta^2} = \frac{i-\delta}{\delta^2},$$

where $\delta = \log_e(1+i) = -\log_e v$.

The two expressions we have found for $\int_0^1 xv^{x-1}dx$ may be shown as follows to be identical.

We have $1+i=e^\delta$

and $$i = e^\delta - 1 = \delta + \frac{\delta^2}{2} + \frac{\delta^3}{6} + \frac{\delta^4}{24} + \frac{\delta^5}{120} + \dots \quad (3)$$

whence $$\frac{i-\delta}{\delta^2} = \frac{1}{2} + \frac{\delta}{6} + \frac{\delta^2}{24} + \frac{\delta^3}{120} + \dots$$

But $$\delta = \log_e(1+i) = i - \frac{i^2}{2} + \frac{i^3}{3} - \frac{i^4}{4} - \dots \quad (4)$$

and substituting this value for δ, we find

$$\frac{i-\delta}{\delta^2} = \frac{1}{2} + \frac{i}{6} - \frac{i^2}{24} + \frac{i^3}{45} - \dots$$

which agrees with (2).

Thus then the value of the correction is

$$A_k\left(\frac{1}{2} + \frac{i}{6} - \frac{i^2}{24} + \frac{i^3}{45} - \dots\right) \quad . \quad . \quad . \quad . \quad (5)$$

This expression agrees with the statement of Professor De Morgan, *Essay on Probabilities*, Appendix the second, p. xxii.—"When an annuity is granted upon condition that the executors of the party are to receive such a proportion of payment for the year in which the annuitant dies, as corresponds to the portion of the year during which he is alive, the addition to the value of the annuity is the present value of $\frac{1}{2} + \frac{i}{6}$ of a year's purchase, payable at the end of the year of death."

As a first approximation to the value of this correction, we may say that on the average £½ will be received in the middle of the year in which death takes place, and the value of the correction will therefore be nearly equal to

$$\frac{A_k}{2}\sqrt{1+i} \text{ or } A_k\left(\frac{1}{2} + \frac{i}{4} - \frac{i^2}{16} + \dots\right)$$

This result, it will be observed, is too large, and is not correct even in the term involving the first power of i.

Baily gives the value of the correction as simply $\frac{A_k}{2}$.

Griffith Davies has given (p. 336, cor. 1) a formula for the value of the correction which is equivalent to

$$\frac{i-\log(1+i)}{i^2}(1-ia_k)$$

$$=\frac{1+i}{i^2}\{i-\log(1+i)\}\mathrm{A}_k$$

$$=(1+i)\left(\frac{1}{2}-\frac{i}{3}+\frac{i^2}{4}-\frac{i^3}{5}+\ldots\right)\mathrm{A}_k$$

$$=\left(\frac{1}{2}+\frac{i}{6}-\frac{i^2}{24}+\frac{i^3}{20}-\ldots\right)\mathrm{A}_k,$$

which agrees with our formula as far as the term involving the first power of i—the remaining terms being different on account of the different supposition he has made as to the interest for fractions of a year.

If we next suppose the annuity to be payable half-yearly instead of yearly, a little consideration will show that the value of the possible payments in the $(t+1)$th year is

$$v^t(p_{k,t}-p_{k,t+1})(1+\sqrt{v})\int_0^{\frac{1}{2}}xv^x dx;$$

and the value of the correction is seen to be, reasoning as before,

$$\mathrm{A}_k\cdot\frac{1+\sqrt{v}}{v}\int_0^{\frac{1}{2}}xv^x dx$$

which, by (2*)

$$=\mathrm{A}_k\frac{1+\sqrt{v}}{v}\left(\frac{1}{\delta^2}-\frac{\sqrt{v}}{2\delta}-\frac{\sqrt{v}}{\delta^2}\right)$$

$$=\mathrm{A}_k\frac{1+\sqrt{v}}{v}\left(\frac{1-\sqrt{v}}{\delta^2}-\frac{\sqrt{v}}{2\delta}\right)$$

$$=\mathrm{A}_k\left(\frac{1-v}{v\delta^2}-\frac{v^{-\frac{1}{2}}+1}{2\delta}\right)$$

$$=\mathrm{A}_k\left(\frac{i}{\delta^2}-\frac{v^{-\frac{1}{2}}+1}{2\delta}\right).$$

Now $\delta=-\log v$, and $v=e^{-\delta}$. Hence

$$v^{-\frac{1}{2}}=e^{\frac{\delta}{2}}=1+\frac{\delta}{2}+\frac{\delta^2}{4}\cdot\frac{1}{2}+\frac{\delta^3}{8}\cdot\frac{1}{6}+\frac{\delta^4}{16}\cdot\frac{1}{24}+\ldots.$$

$$\frac{v^{-\frac{1}{2}}+1}{2\delta}=\frac{1}{\delta}+\frac{1}{4}+\frac{\delta}{16}+\frac{\delta^2}{96}+\frac{\delta^3}{768}+\ldots.$$

Also $$\frac{i}{\delta^2}=\frac{1}{\delta}+\frac{1}{2}+\frac{\delta}{6}+\frac{\delta^2}{24}+\frac{\delta^3}{120}+\ldots,\text{ by (3),}$$

$$\therefore \qquad \frac{i}{\delta^2}-\frac{v^{-\frac{1}{2}}+1}{2\delta}=\frac{1}{4}+\frac{5}{48}\delta+\frac{\delta^2}{32}+\frac{9}{1280}\delta^3+\dots.$$

and the value of the correction is

$$\mathrm{A}_k\left(\frac{1}{4}+\frac{5}{48}\delta+\frac{\delta^2}{32}+\frac{9}{1280}\delta^3+\dots.\right)$$

$$=\mathrm{A}_k\left(\frac{1}{4}+\frac{5}{48}i-\frac{i^2}{48}+\frac{121}{11520}i^3-\dots.\right)\quad . \quad . \quad (6)$$

This result may also be arrived at by finding $\int_0^{\frac{1}{2}} xv^x dx$ from the formula (1). Thus,

$$\int_0^{\frac{1}{2}} xv^x dx=\frac{1}{8}+\frac{i}{12}-\frac{5}{384}i^2+\frac{17}{2880}i^3-\dots.$$

Again, $\qquad 1+\sqrt{v}=1+(1+i)^{-\frac{1}{2}}$

$$=2-\frac{i}{2}+\frac{3}{8}i^2-\frac{5}{16}i^3+\dots.$$

and multiplying these series together, we get the above expression (6) for the value of the correction.

Let the annuity now be supposed to be payable m times a year; and let each year, as the $(t+1)$th, be divided into m equal parts. Then if death occur in the $(s+1)$th part of the $(t+1)$th year, at the instant dx following the time $t+\dfrac{s}{m}+x$, there is payable the sum £x. The chance of death occurring in that instant on the supposition of uniform distribution of deaths is, as before, $(p_{k,t}-p_{k,t+1})dx$; and the value of the chance of receiving £x by such death is

$$xv^{t+\frac{s}{m}+x}(p_{k,t}-p_{k,t+1})dx.$$

The value of all the possible payments in the $(s+1)$th portion of the $(t+1)$th year will be found by integration, and is

$$v^{t+\frac{s}{m}}(p_{k,t}-p_{k,t+1})\int_0^{\frac{1}{m}} xv^x dx.$$

The value of all the possible payments in the $(t+1)$th year will be found by summing with respect to s, giving it the values $0, 1 \dots m-1$; and is therefore equal to

$$v^t(1+v^{\frac{1}{m}}+v^{\frac{2}{m}}+\dots.+v^{\frac{m-1}{m}})(p_{k,t}-p_{k,t+1})\int_0^{\frac{1}{m}} xv^x dx$$

$$=v^t\frac{1-v}{1-v^{\frac{1}{m}}}(p_{k,t}-p_{k,t+1})\int_0^{\frac{1}{m}} xv^x dx.$$

Lastly, summing with respect to t, the total value of the correction is

$$\frac{1-v}{1-v^{\frac{1}{m}}}\cdot\frac{A_k}{v}\int_0^{\frac{1}{m}} xv^x dx=\frac{iA_k}{1-v^{\frac{1}{m}}}\int_0^{\frac{1}{m}} xv^x dx.$$

But, as we have already shown,

$$\int xv^x dx=C-\frac{xv^x}{\delta}-\frac{v^x}{\delta^2}$$

$$\therefore \int_0^{\frac{1}{m}} xv^x dx=\frac{1}{\delta^2}-\frac{v^{\frac{1}{m}}}{m\delta}-\frac{v^{\frac{1}{m}}}{\delta^2}$$

$$=\frac{1-v^{\frac{1}{m}}}{\delta^2}-\frac{v^{\frac{1}{m}}}{m\delta},$$

so that the value of the correction is

$$iA_k\left\{\frac{1}{\delta^2}-\frac{v^{\frac{1}{m}}}{1-v^{\frac{1}{m}}}\cdot\frac{1}{m\delta}\right\}$$

$$=iA_k\left\{\frac{1}{\delta^2}-\frac{1}{v^{-\frac{1}{m}}-1}\cdot\frac{1}{m\delta}\right\}$$

$$=iA_k\left\{\frac{1}{\delta^2}-\frac{1}{e^{\frac{\delta}{m}}-1}\cdot\frac{1}{m\delta}\right\}$$

$$=iA_k\left\{\frac{1}{\delta^2}-\frac{1}{m\delta}\left(\frac{m}{\delta}-\frac{1}{2}+\frac{\delta}{12m}-\frac{1}{720}\frac{\delta^3}{m^3}+\ldots\right)\right\}^*$$

$$=iA_k\left\{\frac{1}{2m\delta}-\frac{1}{12m^2}+\frac{\delta^2}{720m^4}-\ldots\right\}$$

which becomes, since $\frac{i}{\delta}=1+\frac{\delta}{2}+\frac{\delta^2}{6}+\frac{\delta^3}{24}+\ldots$

$$=1+\frac{i}{2}-\frac{i^2}{12}+\frac{i^3}{24}-\ldots$$

$$A_k\left\{\frac{1}{2m}+\frac{i}{4m}\left(1-\frac{1}{3m}\right)-\frac{i^2}{24m}+\frac{i^3}{48m}\left(1+\frac{1}{15m^3}\right)-\ldots\right\} \quad . \quad . \quad (7)$$

Making $m=1, 2, 4$, successively, the values of the corrections in the cases of annuities payable yearly, half-yearly, and quarterly, respectively, are found to be

$$A_k\left(\frac{1}{2}+\frac{i}{6}-\frac{i^2}{24}+\frac{i^3}{45}-\ldots\right)$$

* This step is obtained from the well known expansion of $\frac{t}{e^t-1}$.

$$A_k\left(\frac{1}{4}+\frac{5}{48}i-\frac{i^2}{48}+\frac{121}{11520}i^3-\ldots\right)$$
$$A_k\left(\frac{1}{8}+\frac{11}{192}i-\frac{i^2}{96}+\frac{961}{184320}i^3-\ldots\right)$$

the two first of which agree with the formulæ we have already found above.

The preceding formulæ are analagous to those obtained by Baily for the value of annuities payable half-yearly and quarterly, on the hypothesis of a uniform distribution of the deaths in each year. But we have seen in the case of those annuities that the results obtained by the use of this hypothesis are not to be relied on when accuracy is desired; and it will therefore be desirable to investigate the more accurate formulæ corresponding to Mr. Woolhouse's formula for the value of an annuity payable half-yearly.

Before doing this, however, it will be convenient to investigate the value of an ordinary annuity of which the first payment, instead of being made at the end of a year, is made sooner, say at the end of $\frac{1}{m}$th part of a year.

In this investigation it is almost necessary, and is, at all events, highly desirable, to denote by a suitable symbol the value of the annuity in question. But when we seek for such a symbol, it becomes apparent that no convenient modification of the notation (a_k) for an ordinary annuity can be introduced; and that we must adopt a symbol to denote the value of an annuity-due, *i.e.*, of an annuity payable in advance. Mr. Scratchley has used the symbol a, and Mr. Filipowski, $\acute{a}$, for this purpose. I propose to adopt the symbol $\mathbf{a}$, as being more simple than the latter, and more distinctive than the former. Thus, then,

$$\mathbf{a}_k=1+a_k.$$

Next, suppose this annuity-due to be deferred n years, then $\mathbf{a}_{k\lceil\overline{n}}$ will denote an annuity of which the first payment is made at the end of n years; or

$$\mathbf{a}_{k\lceil\overline{n}}=a_{k|\overline{n-1}}.$$

Also, ${}_{\overline{t}|}\mathbf{a}_k$ will denote a temporary annuity of t payments, the first made at once; or

$${}_{\overline{t}|}\mathbf{a}_k=1+{}_{\overline{t-1}|}a_k.$$

So, again, $${}_{\overline{t}|}\mathbf{a}_{k\lceil\overline{n}}={}_{\overline{t}|}a_{k|\overline{n-1}}.$$

Making $n=0$, we see also that $a_{k|\overline{-1}}$ must be interpreted to mean $\mathbf{a}_k$.

Consistently with this notation, the value of an annuity of

which the first payment is made at the end of the mth part of a year will be denoted by $\mathbf{a}_{k|\overline{\frac{1}{m}}}$.

We have for the value of this annuity

$$\mathbf{a}_{k|\overline{\frac{1}{m}}}=p_{k,\frac{1}{m}}v^{\frac{1}{m}}+p_{k,1+\frac{1}{m}}v^{1+\frac{1}{m}}+p_{k,2+\frac{1}{m}}v^{2+\frac{1}{m}}+\dots$$

$$=\frac{l_{k+\frac{1}{m}}v^{k+\frac{1}{m}}+l_{k+1+\frac{1}{m}}v^{k+1+\frac{1}{m}}+l_{k+2+\frac{1}{m}}v^{k+2+\frac{1}{m}}+\dots}{l_kv^k};$$

or, putting u_x for l_xv^x,

$$\mathbf{a}_{k|\overline{\frac{1}{m}}}=\frac{u_{k+\frac{1}{m}}+u_{k+1+\frac{1}{m}}+u_{k+2+\frac{1}{m}}+\dots}{u_k}$$

Now substituting $\frac{1}{m}$ for x in the formula (8) for u_{k+x} given in my paper in the last Number of this *Journal*, p. 320, we have, very approximately,

$$u_{k+\frac{1}{m}}=u_k\left(1-\frac{3}{m^2}+\frac{2}{m^3}\right)+u_{k+1}\left(\frac{3}{m^2}-\frac{2}{m^3}\right)$$

$$+u'_k\left(\frac{1}{m}-\frac{2}{m^2}+\frac{1}{m^3}\right)+u'_{k+1}\left(-\frac{1}{m^2}+\frac{1}{m^3}\right)$$

$$=u_k-\left(\frac{3}{m^2}-\frac{2}{m^3}\right)(u_k-u_{k+1})+\frac{u'_k}{m}\left(1-\frac{1}{m}\right)^2-\frac{u'_{k+1}}{m^2}\left(1-\frac{1}{m}\right)$$

So also,

$$u_{k+1+\frac{1}{m}}=u_{k+1}-\left(\frac{3}{m^2}-\frac{2}{m^3}\right)(u_{k+1}-u_{k+2})+\frac{u'_{k+1}}{m}\left(1-\frac{1}{m}\right)^2-\frac{u'_{k+2}}{m^2}\left(1-\frac{1}{m}\right)$$

$$u_{k+2+\frac{1}{m}}=u_{k+2}-\&c.$$

$$\&c. = \&c.$$

Adding these equations together, we get

$$u_{k+\frac{1}{m}}+u_{k+1+\frac{1}{m}}+u_{k+2+\frac{1}{m}}+\dots\dots=\Sigma u_{k+\frac{1}{m}}$$

$$=u_k+u_{k+1}+u_{k+2}+\dots\dots-\left(\frac{3}{m^2}-\frac{2}{m^3}\right)u_k$$

$$+\frac{1}{m}\left(1-\frac{1}{m}\right)^2(u'_k+u'_{k+1}+u'_{k+2}+\dots)$$

$$-\frac{1}{m^2}\left(1-\frac{1}{m}\right)(u'_{k+1}+u'_{k+2}+\dots)$$

$$=u_k+u_{k+1}+u_{k+2}+\dots\dots-\left(\frac{3}{m^2}-\frac{2}{m^3}\right)u_k$$

$$+\frac{1}{m}\left(1-\frac{1}{m}\right)^2u'_k+\frac{1}{m}\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)(u'_{k+1}+u'_{k+2}+\dots)$$

But applying the formula (p. 207)

$$\frac{u_a}{2} + u_{a+1} + u_{a+2} + \dots\dots + \frac{u_b}{2} = \int_a^b u_x dx + \frac{1}{12}(u'_b - u'_a)$$
$$- \frac{1}{720}(u'''_b - u'''_a)$$
$$+ \quad . \quad . \quad . \quad .$$

we have

$$\frac{u'_k}{2} + u'_{k+1} + u'_{k+2} + \dots + \frac{u'_z}{2} = -u_k + \frac{u''_z - u''_k}{12} - \frac{u_z^{\mathrm{IV}} - u_k^{\mathrm{IV}}}{720} + \dots$$

—z being used (as at p. 317) to denote the extreme age in the table of mortality, which none of the lives actually attain.

The first term in the second member of this equation is obtained thus:

$$\int u'_x dx = u_x + \mathrm{C},$$

$$\therefore \qquad \int_k^z u'_x dx = u_z - u_k = -u_k, \quad \text{since } u_z = 0.$$

The quantities u'_z, u''_z, u_z^{IV}. . . . are generally so small in comparison with the others that they may be neglected, and we have therefore very approximately

$$\Sigma u_{k+\frac{1}{m}} = \Sigma u_k - \left(\frac{3}{m^2} - \frac{2}{m^3}\right) u_k + \frac{1}{m}\left(1 - \frac{1}{m}\right)^2 u'_k$$
$$+ \frac{1}{m}\left(1 - \frac{1}{m}\right)\left(1 - \frac{2}{m}\right)\left(-u_k - \frac{u'_k}{2} - \frac{u''_k}{12}\right)$$
$$= \Sigma u_k - \frac{1}{m} u_k + \frac{1}{2m}\left(1 - \frac{1}{m}\right) u'_k - \frac{1}{12m}\left(1 - \frac{1}{m}\right)\left(1 - \frac{2}{m}\right) u''_k.$$

Dividing both sides by u_k, we get

$$\mathbf{a}_{k\overline{\left|\frac{1}{m}\right.}} = 1 + a_k - \frac{1}{m} + \frac{m-1}{2m^2} \cdot \frac{u'_k}{u_k} - \frac{(m-1)(m-2)}{12m^3} \cdot \frac{u''_k}{u_k}.$$

But since $u_k = l_k v^k$,

$$u'_k = l'_k v^k + \log v \, l_k v^k,$$

and $$\frac{u'_k}{u_k} = \frac{l'_k - \delta l_k}{l_k} = -\mu - \delta.$$

Also, $$u''_k = l''_k v^k - 2\delta l'_k v^k + \delta^2 l_k v^k,$$

and $$\frac{u''_k}{u_k} = \frac{l''_k}{l_k} + 2\mu\delta + \delta^2;$$

whence, by substitution,

$$\mathbf{a}_{k\overline{\left|\frac{1}{m}\right.}}=a_k+\frac{m-1}{m}-\frac{m-1}{2m^2}(\mu+\delta)-\frac{(m-1)(m-2)}{12m^3}\left(\frac{l''_k}{l_k}+2\mu\delta+\delta^2\right)\,..\,(8)$$

very approximately.

Or, adopting the notation I have employed in the last Number (p. 317),

$$\mathbf{a}_{k\overline{\left|\frac{1}{m}\right.}}=a_k+\frac{m-1}{m}-\frac{m-1}{2m^2}(\mu+\delta)-\frac{(m-1)(m-2)}{12m^3}\frac{D''_k}{D_k}\quad . \quad . \quad . \quad (9)$$

The most important case is when $m=2$, or the first payment of the annuity is made at the end of six months. In that case,

$$\mathbf{a}_{k\overline{\left|\frac{1}{2}\right.}}=a_k+\tfrac{1}{2}-\frac{\mu+\delta}{8}\quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

the last term in the value of $\mathbf{a}_{k\overline{\left|\frac{1}{m}\right.}}$ disappearing.

The universal practice of actuaries is to take the value of such an annuity as $a_k+\frac{1}{2}$; and we now see that that value is too large. The error, however, is seldom of much practical importance. Taking an average age, 45, and interest at three per cent., we have, according to the Experience table, $\frac{\mu+\delta}{8}=\cdot0052$; and if we are valuing the liabilities of an Office in which the premiums amount to £100,000 a year, the error caused by using the common approximation will only amount to £520.

In order to test the accuracy of the above formula (9) we will apply it to find the value of an annuity payable m times a year. Writing $\frac{m}{r}$ for m, we have

$$\mathbf{a}_{k\overline{\left|\frac{r}{m}\right.}}=a_k+\frac{m-r}{m}-\frac{r(m-r)}{2m^2}(\mu+\delta)-\frac{r(m-r)(m-2r)}{12m^3}\frac{D''_k}{D_k}.$$

Here making r equal to 1, 2, 3 . . . m, successively, we have, by addition,

$$\mathbf{a}_{k\overline{\left|\frac{1}{m}\right.}}+\mathbf{a}_{k\overline{\left|\frac{2}{m}\right.}}+\mathbf{a}_{k\overline{\left|\frac{3}{m}\right.}}+\,.\,.\,.\,.\,.\,.\,+\mathbf{a}_{k\overline{|1}}$$

$$=ma_k+m-\frac{1}{m}\Sigma r-\frac{\mu+\delta}{2m^2}\{m\Sigma r-\Sigma r^2\}$$

$$-\frac{D''_k}{12m^3D_k}\{m^2\Sigma r-3m\Sigma r^2+2\Sigma r^3\}.$$

But the first member of this equation is equal to $ma_k^{(m)}$.

Also $\Sigma r=1+2+3+\ldots.+m=\dfrac{m(m+1)}{2}$

$$\Sigma r^2=1+2^2+3^3+\ldots+m^2=\frac{m(m+1)(2m+1)}{6}$$

$$\Sigma r^3=1+2^3+3^3+\ldots.+m^3=\frac{m^2(m+1)^2}{4}$$

Hence we have

$$ma_k^{(m)}=ma_k+m-\frac{m+1}{2}-\frac{m+1}{4m}(\mu+\delta)\left\{m-\frac{2m+1}{3}\right\}$$

$$-\frac{D''_k}{D_k}\cdot\frac{m+1}{24m^2}\{m^2-m(2m+1)+m(m+1)\}$$

$$=ma_k+\frac{m-1}{2}-\frac{m^2-1}{12m}(\mu+\delta),$$

the other term disappearing;

and $a_k^{(m)}=a_k+\dfrac{m-1}{2m}-\dfrac{m^2-1}{12m^2}(\mu+\delta),$

which is Mr. Woolhouse's formula. It will be noticed that in deducing this from our formula for $\mathbf{a}_{k\overline{\left|\frac{r}{m}\right.}}$, no process of approximation has been employed; but if the one formula were strictly accurate, so also would the other be.

The above formula for $\mathbf{a}_{k\overline{\left|\frac{1}{m}\right.}}$ needs no further confirmation. I subjoin, however, another demonstration, as illustrating a different mode of proceeding which may be found useful in other investigations.

We have, u_x being any function of x,

$$\frac{u_a}{2}+u_{a+h}+u_{a+2h}+\ldots.+u_{b-h}+\frac{u_b}{2}$$

$$=\frac{1}{h}\int_a^b u_x dx+\frac{h}{12}(u'_b-u'_a)-\frac{h^3}{720}(u'''_b-u'''_a)+\ \ .\ \ .\ \ (11)$$

Now put u_x equal to l_xv^x; and first let $a=k$, $h=1$, $b=z$, the extreme age in the table of mortality, which none of the lives actually attain, then the first member of (11) becomes $l_kv^k(\frac{1}{2}+a_k)$; and we get

$$l_kv^k(\tfrac{1}{2}+a_k)=\int_k^z u_x dx-\frac{1}{12}(u'_k-u'_z)+\frac{1}{720}(u'''_k-u'''_z)-\ldots.$$

$$=\int_k^z u_x dx-\frac{u'_k}{12}+\frac{u'''}{720}-\ldots\ \ .\ \ .\ \ .\ \ .\ \ (12)$$

approximately, the values u'_z, u'''_z, . . . being very small.

Next, put $a=k+\frac{1}{m}$, $h=1$, b corresponding, as before, to the extremity of the mortality table; then we get

$$l_kv^k.\mathbf{a}_{k\overline{\left|\frac{1}{m}\right.}}-\tfrac{1}{2}l_{k+\frac{1}{m}}v^{k+\frac{1}{m}}=\int_{k+\frac{1}{m}}^{z}u_xdx-\frac{1}{12}u'_{k+\frac{1}{m}}+\frac{1}{720}u'''_{k+\frac{1}{m}}-\ldots \quad (13)$$

Lastly, put $a=k$, $h=\frac{1}{m}$, $b=a+h=k+\frac{1}{m}$, then the first member of (11) reduces to $\frac{u_a+u_b}{2}$ or $\frac{1}{2}(u_k+u_{k+\frac{1}{m}})$; and we have

$$\tfrac{1}{2}(l_kv^k+l_{k+\frac{1}{m}}v^{k+\frac{1}{m}})=m\int_k^{k+\frac{1}{m}}u_xdx+\frac{1}{12m}(u'_{k+\frac{1}{m}}-u'_k)$$
$$-\frac{1}{720m^3}(u'''_{k+\frac{1}{m}}-u'''_k)$$
$$+\ldots\ldots$$

Dividing this by m, and adding it to (13), we get, since

$$\int_k^{k+\frac{1}{m}}u_xdx+\int_{k+\frac{1}{m}}^{z}u_xdx=\int_k^z u_xdx,$$

$$l_kv^k\left(\mathbf{a}_{k\overline{\left|\frac{1}{m}\right.}}+\frac{1}{2m}\right)-\tfrac{1}{2}\left(1-\frac{1}{m}\right)l_{k+\frac{1}{m}}v^{k+\frac{1}{m}}=\int_k^z u_xdx$$
$$-\frac{1}{12m^2}\{(m^2-1)u'_{k+\frac{1}{m}}+u'_k\}$$
$$+\frac{1}{720m^4}\{(m^4-1)u'''_{k+\frac{1}{m}}+u'''_k\}$$
$$-\ldots\ldots\ldots$$

Now subtracting (12) from this, we have

$$l_kv^k\left\{\mathbf{a}_{k\overline{\left|\frac{1}{m}\right.}}-a_k-\tfrac{1}{2}\left(1-\frac{1}{m}\right)\right\}-\tfrac{1}{2}\left(1-\frac{1}{m}\right)l_{k+\frac{1}{m}}v^{k+\frac{1}{m}}$$
$$=\frac{m^2-1}{12m^2}(u'_k-u'_{k+\frac{1}{m}})-\frac{m^4-1}{720m^4}(u'''_k-u'''_{k+\frac{1}{m}})+\ldots.$$

whence

$$\mathbf{a}_{k\overline{\left|\frac{1}{m}\right.}}=a_k+\frac{m-1}{2m}\left(1+\frac{l_{k+\frac{1}{m}}}{l_k}\cdot v^{\frac{1}{m}}\right)+\frac{m^2-1}{12m^2}\cdot\frac{u'_k-u'_{k+\frac{1}{m}}}{l_kv^k}$$
$$-\frac{m^4-1}{720m^4}\cdot\frac{u'''_k-u'''_{k+\frac{1}{m}}}{l_kv^k}+\ldots.$$

But since $u_x=l_xv^x$,

$$u'_x=l'_xv^x-\delta l_xv^x,$$

so that

$$u'_k=l'_kv^k-\delta l_kv^k,$$

and $$u'_{k+\frac{1}{m}}=(l'_{k+\frac{1}{m}}-\delta l_{k+\frac{1}{m}})v^{k+\frac{1}{m}}$$

$$=\left\{l'_k+\frac{1}{m}l''_k+\ldots.-\delta\left(l_k+\frac{l'_k}{m}+\frac{l''_k}{2m^2}+\ldots\right)\right\}v^{k+\frac{1}{m}}$$

(by Taylor's theorem)

$$=\left\{-\delta l_k+\left(1-\frac{\delta}{m}\right)l'_k+\frac{1}{m}\left(1-\frac{\delta}{2m}\right)l''_k+\ldots\right\}v^{k+\frac{1}{m}},$$

whence $$u'_k-u'_{k+\frac{1}{m}}=-\delta l_k v^k(1-v^{\frac{1}{m}})+l'_k v^k\left\{1-v^{\frac{1}{m}}+\frac{\delta}{m}v^{\frac{1}{m}}\right\}$$

$$-\frac{1}{m}\left(1-\frac{\delta}{2m}\right)l''_k v^{k+\frac{1}{m}}$$

$$+\ \ldots\ldots$$

Also $$\frac{l_{k+\frac{1}{m}}}{l_k}=\frac{1}{l_k}\left\{l_k+\frac{1}{m}l'_k+\frac{1}{2m^2}l''_k+\ldots.\right\}$$

$$=1-\frac{\mu}{m}+\frac{1}{2m^2}\frac{l''_k}{l_k}+\ldots.$$

We therefore have

$$\mathbf{a}_{k\,\overline{|\tfrac{1}{m}}}=a_k+\frac{m-1}{2m}\left\{1+\left(1-\frac{\mu}{m}+\frac{1}{2m^2}\frac{l''_k}{l_k}\right)v^{\frac{1}{m}}\right\}$$

$$+\frac{m^2-1}{12m^2}\left\{-\delta(1-v^{\frac{1}{m}})-\mu\left(1-v^{\frac{1}{m}}+\frac{\delta}{m}v^{\frac{1}{m}}\right)-\frac{1}{m}\left(1-\frac{\delta}{2m}\right)\frac{l''_k}{l_k}v^{\frac{1}{m}}\right\}$$

approximately.

Again, since $\delta=-\log v$, $v=e^{-\delta}$, and

$$v^{\frac{1}{m}}=e^{-\frac{\delta}{m}}=1-\frac{}{m}+\frac{\delta^2}{2m^2}-\ldots.$$

we find, neglecting terms that involve δ^3, $\mu\delta^2$, and $l''_k\delta$,

$$\delta(1-v^{\frac{1}{m}})=\frac{\delta^2}{m},\ \text{approximately;}$$

$$\mu\left(1-v^{\frac{1}{m}}+\frac{\delta}{m}v^{\frac{1}{m}}\right)=\mu\left(\frac{\delta}{m}+\frac{\delta}{m}\right)=2\frac{\delta\mu}{m},\ \text{approximately;}$$

$$\frac{1}{m}\left(1-\frac{\delta}{2m}\right)\frac{l''_k}{l_k}v^{\frac{1}{m}}=\frac{l''_k}{ml_k},\ \text{approximately.}$$

Also

$$1+\left(1-\frac{\mu}{m}+\frac{1}{2m^2}\frac{l''_k}{l_k}\right)v^{\frac{1}{m}}=1+\left(1-\frac{\mu}{m}+\frac{1}{2m^2}\frac{l''_k}{l_k}\right)\left(1-\frac{\delta}{m}+\frac{\delta^2}{2m^2}\right)$$

$$=2-\frac{\mu+\delta}{m}+\frac{\delta^2}{2m^2}+\frac{\mu\delta}{m^2}+\frac{l''_k}{2m^2l_k},\ \text{approximately;}$$

so that, substituting, we get finally

$$\mathbf{a}_{k\overline{\left|\frac{1}{m}\right.}}=a_k+\frac{m-1}{2m}\left\{2-\frac{\mu+\delta}{m}+\frac{\delta^2}{2m^2}+\frac{\mu\delta}{m^2}+\frac{l''_k}{2m^2l_k}\right\}$$

$$+\frac{m^2-1}{12m^2}\left\{-\frac{\delta^2}{m}-\frac{2\delta\mu}{m}-\frac{l''_k}{ml_k}\right\}$$

$$=a_k+\frac{m-1}{m}-\frac{m-1}{2m^2}(\mu+\delta)-\frac{(m-1)(m-2)}{12m^3}\left(\delta^2+2\mu\delta+\frac{l''_k}{l_k}\right)$$

—as found above (8).

As a further test of the accuracy of this formula, apply it to the case of a perpetual annuity certain; and for this purpose make μ and l''_k both zero. Then a_k becomes equal to $\frac{1}{i}$; and the value of the annuity, as found from (8), becomes

$$\frac{1}{i}+\frac{m-1}{m}-\frac{m-1}{2m^2}\delta-\frac{(m-1)(m-2)}{12m^3}\delta^2 \quad . \quad . \quad . \quad (14)$$

But the value of the annuity certain in this case is

$$v^{\frac{1}{m}}+v^{1+\frac{1}{m}}+v^{2+\frac{1}{m}}+\ldots.\text{ad inf.}$$

$$=\frac{v^{\frac{1}{m}}}{1-v}=\frac{1+i}{i}v^{\frac{1}{m}}=\frac{1}{i}v^{\frac{1}{m}-1}=\frac{e^{\left(1-\frac{1}{m}\right)\delta}}{i}=\frac{1}{i}\times e^{\frac{m-1}{m}.\delta}$$

$$=\frac{1}{i}\left\{1+\frac{m-1}{m}\delta+\frac{(m-1)^2}{m^2}\frac{\delta^2}{2}+\frac{(m-1)^3}{m^3}\cdot\frac{\delta^3}{6}+\ldots.\right\}$$

$$=\frac{1}{i}+\frac{m-1}{m}\frac{\delta}{i}\left\{1+\frac{m-1}{m}\frac{\delta}{2}+\frac{(m-1)^2}{m^2}\frac{\delta^2}{6}+\ldots.\right\}$$

$$=\frac{1}{i}+\frac{m-1}{m}-\frac{m-1}{2m^2}\delta-\frac{(m-1)(m-2)}{12m^3}\delta^2+\ldots.$$

—which is the same as found above (14).

For $\quad v=\frac{1}{1+i}=e^{-\delta} \quad \therefore\ 1+i=e^{\delta}, \quad$ and $\quad i=e^{\delta}-1,$

$$\therefore \qquad \frac{\delta}{i}=\frac{\delta}{e^{\delta}-1}=1-\frac{\delta}{2}+\frac{\delta^2}{12}-\frac{\delta^4}{720}+\ldots\ldots \quad (15)$$

this being the well known expansion by means of Bernouilli's numbers given in all treatises on the Calculus of Finite Differences.

We now return to the investigation of the increase in the value of an annuity, when upon the occurrence of death a further payment is made proportioned to the time which has elapsed since the last payment of the annuity. This increase will be called, as before, the "correction."

First, let the annuity be supposed to be payable yearly.

Then, the present age being k, if death occur in the $(t+1)$th year from the present time—say in the instant dx following the time x in that year; *i.e.*, at the time $t+x$ from the present time, there will be payable a sum £x. The chance of death occurring as supposed will be

$$\frac{l_{k+t+x}-l_{k+t+x+dx}}{l_k}=-\frac{l'_{k+t+x}}{l_k}\cdot dx,$$

and the value of £x to be received on the occurrence of such death will be

$$-xv^{t+x}\cdot\frac{l'_{k+t+x}}{l_k}\cdot dx \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

and the value of the correction will be found by integrating this quantity with respect to x between the values 0 and 1; and then summing it with respect to t, giving t the values 0, 1, 2 .... to the extremity of life. Now we have, using the formula of interpolation employed above,

$$l_{k+t+x}=l_{k+t}-(3x^2-2x^3)(l_{k+t}-l_{k+t+1})$$
$$+l'_{k+t}(x-2x^2+x^3)-l'_{k+t+1}(x^2-x^3) \quad . \quad . \quad (17)$$

Hence

$$l'_{k+t+x}=-6(x-x^2)(l_{k+t}-l_{k+t+1})+l'_{k+t}(1-4x+3x^2)-l'_{k+t+1}(2x-3x^2),$$

and substituting in (16) this value of l'_{k+t+x}, we have first to find the value of

$$\int_0^1\frac{v^{t+x}}{l_k}\left\{6(x^2-x^3)(l_{k+t}-l_{k+t+1})-l'_{k+t}(x-4x^2+3x^3)\right.$$
$$\left.+l'_{k+t+1}(2x^2-3x^3)\right\}dx \quad . \quad . \quad (18)$$

Now $\qquad \delta=-\log_e v \quad \therefore \quad v=e^{-\delta}$

and $\qquad v^x=e^{-\delta x}=1-\delta x+\frac{\delta^2x^2}{2}-\frac{\delta^3x^3}{6}+\ldots.$

whence $\displaystyle\int x^nv^xdx=\int\left(x^n-\delta x^{n+1}+\frac{\delta^2}{2}x^{n+2}-\frac{\delta^3}{6}x^{n+3}+\ldots.\right)dx$

$$=\frac{x^{n+1}}{n+1}-\frac{\delta x^{n+2}}{n+2}+\frac{\delta^2x^{n+3}}{2(n+3)}-\frac{\delta^3x^{n+4}}{6(n+4)}+\ldots.+C,$$

and $\int_0^1 x^n v^x dx = \frac{1}{n+1} - \frac{\delta}{n+2} + \frac{\delta^2}{2(n+3)} - \frac{\delta^3}{6(n+4)} + \ldots.$

Here making $n=1, 2, 3$, successively, we get

$$\int_0^1 x\, v^x dx = \tfrac{1}{2} - \frac{\delta}{3} + \frac{\delta^2}{8} - \frac{\delta^3}{30} + \ldots.$$

$$\int_0^1 x^2 v^x dx = \tfrac{1}{3} - \frac{\delta}{4} + \frac{\delta^2}{10} - \frac{\delta^3}{36} + \ldots.$$

$$\int_0^1 x^3 v^x dx = \tfrac{1}{4} - \frac{\delta}{5} + \frac{\delta^2}{12} - \frac{\delta^3}{42} + \ldots.$$

Hence $\int_0^1 (x^2 - x^3) v^x dx = \tfrac{1}{12} - \frac{\delta}{20} + \frac{\delta^2}{60} - \frac{\delta^3}{252} + \ldots.$

$$\int_0^1 (x - 4x^2 + 3x^3) dx = -\tfrac{1}{12} + \frac{\delta}{15} - \frac{\delta^2}{40} + \frac{2}{315}\delta^3 - \ldots.$$

$$\int_0^1 (2x^2 - 3x^3) v^x dx = -\tfrac{1}{12} + \frac{\delta}{10} - \frac{\delta^2}{20} + \frac{\delta^3}{63} - \ldots.$$

and substituting these values in (18), we get the value of the correction in the $(t+1)$th year, equal to

$$-\frac{v^t}{l_k}\int_0^1 x v^x l'_{k+t+x} dx = \frac{v^t}{l_k}\Big\{6\Big(\tfrac{1}{12} - \frac{\delta}{20} + \frac{\delta^2}{60} - \frac{\delta^3}{252} + \ldots\Big)(l_{k+t} - l_{k+t+1})$$
$$+ \Big(\tfrac{1}{12} - \frac{\delta}{15} + \frac{\delta^2}{40} - \frac{2}{315}\delta^3 + \ldots\Big) l'_{k+t}$$
$$- \Big(\tfrac{1}{12} - \frac{\delta}{10} + \frac{\delta^2}{20} - \frac{\delta^3}{63} + \ldots\Big) l'_{k+t+1}\Big\}.$$

Next summing with regard to t, giving it the values 0, 1, 2... we get the total value of the correction, neglecting terms involving higher powers than δ^3,

$$\left.\begin{aligned} &= \Big(\tfrac{1}{2} - \frac{3}{10}\delta + \frac{\delta^2}{10} - \frac{\delta^3}{42}\Big)\frac{1}{l_k}\Sigma v^t(l_{k+t} - l_{k+t+1}) \\ &\quad + \Big(\tfrac{1}{12} - \frac{\delta}{15} + \frac{\delta^2}{40} - \frac{2}{315}\delta^3\Big)\frac{1}{l_k}\Sigma v^t l'_{k+t} \\ &\quad - \Big(\tfrac{1}{12} - \frac{\delta}{10} + \frac{\delta^2}{20} - \frac{\delta^3}{63}\Big)\frac{1}{l_k}\Sigma v^t l'_{k+t+1} \end{aligned}\right\} \quad \ldots \quad (19)$$

Now we notice that

$$\Sigma v^{t+1}\frac{l_{k+t} - l_{k+t+1}}{l_k} = A_k$$

—the value of an assurance on the life k. But we shall find it more convenient to obtain an expression involving the annuity, a_k.

Thus $$\Sigma v^t \frac{l_{k+t}}{l_k} = 1 + a_k,$$

and $$\Sigma v^t \frac{l_{k+t+1}}{l_k} = \frac{1}{v}a_k = \left(1+\delta+\frac{\delta^2}{2}+\frac{\delta^3}{6}\right)a_k,$$

so that

$$\left(\tfrac{1}{2}-\frac{3}{10}\delta+\frac{\delta^2}{10}-\frac{\delta^3}{42}\right)\frac{1}{l_k}\Sigma v^t(l_{k+t}-l_{k+t+1})$$

$$=\left(\tfrac{1}{2}-\frac{3}{10}\delta+\frac{\delta^2}{10}-\frac{\delta^3}{42}\right)\left\{1+a_k-\left(1+\delta+\frac{\delta^2}{2}+\frac{\delta^3}{6}\right)a_k\right\}$$

$$=\tfrac{1}{2}-\frac{3}{10}\delta+\frac{\delta^2}{10}-\frac{\delta^3}{42}-\left(\frac{\delta}{2}-\frac{\delta^2}{20}+\frac{\delta^3}{30}\right)a_k, \text{ approximately.}$$

Again, $$\Sigma v^t l'_{k+t+1} = l'_{k+1}+vl'_{k+2}+v^2l'_{k+3}+\ldots.$$

$$=\frac{1}{v}\{\Sigma v^t l'_{k+t}-l'_k\}$$

$$=\left(1+\delta+\frac{\delta^2}{2}+\frac{\delta^3}{6}\right)\{\Sigma v^t l'_{k+t}-l'_k\}$$

whence

$$\left(\tfrac{1}{12}-\frac{\delta}{15}+\frac{\delta^2}{40}-\frac{2}{315}\delta^3\right)\frac{1}{l_k}\Sigma v^t l'_{k+t}-\left(\tfrac{1}{12}-\frac{\delta}{10}+\frac{\delta^2}{20}-\frac{\delta^3}{63}\right)\frac{1}{l_k}\Sigma v^t l'_{k+t+1}$$

$$=\left(\tfrac{1}{12}-\frac{\delta}{15}+\frac{\delta^2}{40}-\frac{2}{315}\delta^3\right)\frac{1}{l_k}\Sigma v^t l'_{k+t}$$

$$-\left(\tfrac{1}{12}-\frac{\delta}{60}-\frac{\delta^2}{120}-\frac{\delta^3}{504}\right)\frac{1}{l_k}(\Sigma v^t l'_{k+t}-l'_k)$$

$$=\left(-\frac{\delta}{20}+\frac{\delta^2}{30}-\frac{11}{2520}\delta^3\right)\Sigma\frac{v^t l'_{k+t}}{l_k}$$

$$-\left(\tfrac{1}{12}-\frac{\delta}{60}-\frac{\delta^2}{120}-\frac{\delta^3}{504}\right)\mu,$$

since $\mu = -\dfrac{l'_k}{l_k}$.

It only remains to evaluate $\Sigma v^t l'_{k+t}$

$$=l'_k+vl'_{k+1}+v^2l'_{k+2}+\ldots.$$

Now putting $v^t l'_{k+t}$ for u_t in the equation

$$\frac{u_a}{2}+u_{a+1}+u_{a+2}+\ldots.+\frac{u_b}{2}=\int_a^b u_t dt+\frac{u'_b-u'_a}{12}, \text{ nearly,}$$

and making $a=0$, $b=z-k$, we get

$$\Sigma v^t l'_{k+t}=\frac{l'_k}{2}+\int_0^{z-k}u_t dt-\frac{u'_0}{12}, \text{ very nearly,}$$

u'_{z-k} being so small that it may be neglected.

But
$$\int u_t dt = \int v^t l'_{k+t} dt$$
$$= v^t l_{k+t} + \delta \int v^t l_{k+t} dt$$
—integrating by parts,

and
$$\int_0^{z-k} u_t dt = -l_k + \delta \int_0^{z-k} v^t l_{k+t} dt$$
$$= -l_k + \delta l_k \left(a_k + \tfrac{1}{2} - \frac{\mu+\delta}{12} \right), \text{ very nearly;}$$

since $\int_0^{z-k} \frac{v^t l_{k+t}}{l_k} dt$ is the value of an annuity payable momently.

Also
$$u'_t = -\delta v^t l'_{k+t} + v^t l''_{k+t},$$
and
$$u'_0 = -\delta l'_k + l''_k;$$
so that
$$\Sigma v^t l'_{k+t} = \frac{l'_k}{2} - l_k + \delta l_k \left(a_k + \tfrac{1}{2} - \frac{\mu+\delta}{12} \right) + \frac{\delta l'_k - l''_k}{12},$$
and
$$\Sigma \frac{v^t l'_{k+t}}{l_k} = -\frac{\mu}{2} - 1 + \delta \left(a_k + \tfrac{1}{2} - \frac{\mu+\delta}{12} \right) - \tfrac{1}{12} \left(\mu\delta + \frac{l''_k}{l_k} \right)$$
$$= \delta(a_k + \tfrac{1}{2}) - 1 - \frac{\mu}{2} - \tfrac{1}{12} \left(\delta^2 + 2\mu\delta + \frac{l''_k}{l_k} \right).$$

The value of the correction thus becomes, by successive substitutions in (19),

$$\tfrac{1}{2} - \frac{3}{10}\delta + \frac{\delta^2}{10} - \frac{\delta^3}{42} - \left(\frac{\delta}{2} - \frac{\delta^2}{20} + \frac{\delta^3}{30} \right) a_k$$
$$- \left(\frac{\delta}{20} - \frac{\delta^2}{30} + \frac{11}{2520}\delta^3 \right) \left\{ \delta(a_k + \tfrac{1}{2}) - 1 - \frac{\mu}{2} - \tfrac{1}{12} \left(\delta^2 + 2\mu\delta + \frac{l''_k}{l_k} \right) \right\}$$
$$- \left(\tfrac{1}{12} - \frac{\delta}{60} - \frac{\delta^2}{120} - \frac{\delta^3}{504} \right) \mu.$$

Multiplying out, and neglecting small quantities of the third order, involving δ^3, $\delta^2\mu$, or $\delta l''_k$, retaining however the term involving δ^3 in the coefficient of a_k, we get the value of the correction

$$\tfrac{1}{2} - \frac{3}{10}\delta + \frac{\delta^2}{10} - \left(\frac{\delta}{2} - \frac{\delta^2}{20} + \frac{\delta^3}{30} \right) a_k$$
$$+ \frac{\delta}{20} - \frac{\delta^2}{30} - \left(\frac{\delta^2}{20} - \frac{\delta^3}{30} \right) a_k$$
$$- \frac{\delta^2}{40} + \frac{\mu\delta}{40} - \frac{\mu}{12} + \frac{\mu\delta}{60}$$
$$= \tfrac{1}{2} - \frac{\delta}{4} + \frac{\delta^2}{24} - \frac{\delta}{2} a_k - \frac{\mu}{12} + \frac{\mu\delta}{24}$$
$$= \tfrac{1}{2}(1 - \delta a_k) - \frac{\delta}{4} - \frac{\mu}{12} + \frac{\delta(\mu+\delta)}{24} \quad . \quad . \quad . \quad . \quad (20)$$

Hence, the value of an annuity payable yearly, with a proportionate part payable to the day of death, is

$$a_k+\tfrac{1}{2}-\frac{\delta}{4}+\frac{\delta^2}{24}-\frac{\delta}{2}a_k-\frac{\mu}{12}+\frac{\mu\delta}{24}$$

$$=(a_k+\tfrac{1}{2})\left(1-\frac{\delta}{2}\right)-\frac{\mu}{12}\left(1-\frac{\delta}{2}\right)+\frac{\delta^2}{24}$$

$$=\left(a_k+\tfrac{1}{2}-\frac{\mu}{12}\right)\left(1-\frac{\delta}{2}\right)+\frac{\delta^2}{24}$$

$$=\left(a_k+\tfrac{1}{2}-\frac{\mu+\delta}{12}\right)\left(1-\frac{\delta}{2}\right)+\frac{\delta}{12} \quad . \quad . \quad . \quad . \quad (21)$$

This method will not give us the value of the annuity with greater accuracy; inasmuch as the fundamental equation (17) neglects small quantities of the third order.

When the annuity is payable, not yearly, but m times in the year, the value of the correction may be found as follows:—

Suppose the unit of time is taken to be $\frac{1}{m}$ th part of a year, instead of one year, and that a payment of £1 is made at the expiration of each m-part. Then the amount of £1 at the end of one unit $\left(\frac{1}{m}\text{th of a year}\right)$ will be $(1+i)^{\frac{1}{m}}$; and we must substitute for δ in the equation (20), $\log(1+i)^{\frac{1}{m}}$; *i.e.* $\frac{1}{m}\log(1+i)$, or $\frac{\delta}{m}$. We must also substitute $ma_k^{(m)}$ for a_k. Lastly, with regard to μ; this quantity denoting the instantaneous mortality at the age k, referred to a unit of time (*i.e.* a year), we must substitute for μ, $\frac{\mu}{m}$, since $\frac{1}{m}$ th of a year is now the unit of time. Making these substitutions, the formula (20) becomes

$$\tfrac{1}{2}\left(1-\frac{\delta}{m}\cdot ma_k^{(m)}\right)-\frac{\delta}{4m}-\frac{\mu}{12m}+\frac{\delta(\mu+\delta)}{24m^2}.$$

This applies to the value of an annuity of £m payable by m instalments in each year; and dividing by m, to obtain the proper formula for an annuity of £1, we get the value of the correction

$$\frac{1}{2m}(1-\delta a_k^{(m)})-\frac{\delta}{4m^2}-\frac{\mu}{12m^2}+\frac{\delta(\mu+\delta)}{24m^3}$$

$$\text{or } \frac{1}{2m}\left\{1-\delta\left[a_k+\frac{m-1}{2m}-\frac{m^2-1}{12m^2}(\mu+\delta)\right]\right\}-\frac{\delta}{4m^2}-\frac{\mu}{12m^2}+\frac{\delta(\mu+\delta)}{24m^3}$$

$$=\frac{1}{2m}(1-\delta a_k)-\frac{\delta}{4m}-\frac{\mu}{12m^2}+\frac{\delta(\mu+\delta)}{24m} \quad . \quad . \quad (22)$$

Adding to this the value of $a_k^{(m)}$, we shall find that the value of an annuity payable m times a year, with a proportionate part to the day of death, is

$$\left(a_k+\tfrac{1}{2}-\frac{\mu+\delta}{12}\right)\left(1-\frac{\delta}{2m}\right)+\frac{\delta}{12m^2} \quad . \quad . \quad . \quad (23)$$

If we put $m=1$, these expressions become the same as (20) and (21). If, again, we make m infinite, (23) reduces to

$$a_k+\tfrac{1}{2}-\frac{\mu+\delta}{12}$$

—the value of an annuity payable momently.

The preceding demonstrations being rather intricate, it will be satisfactory to show how the same results may be arrived at by a different process.

I will now take the case of an annuity payable m times a year. The year being, as before, supposed to be divided into m equal parts, let it be further supposed that each m-part is subdivided into n equal portions; and that if death occur in the rth subdivision of any m-part, the sum of $\frac{r}{mn}$ is payable at the end of that subdivision; then when n is supposed to become infinite, we get the case of an annuity payable m times a year with a proportionate part to the day of death. The value of the chance of receiving $\frac{r}{mn}$, as aforesaid, by the occurrence of death in the rth subdivision of the $(s+1)$th part of the $(t+1)$th year, is

$$\frac{r}{mn}\cdot v^{t+\frac{s}{m}+\frac{r}{mn}}\cdot\left\{p_{k,\,t+\frac{s}{m}+\frac{r-1}{mn}}-p_{k,\,t+\frac{s}{m}+\frac{r}{mn}}\right\} \quad . \quad . \quad . \quad . \quad (24)$$

and we have now to sum this quantity with respect to t, s, and r.

It is immaterial which of these variables we deal with first. Summing first with regard to t, and giving it the values 0, 1, 2... to the extremity of life, the sum is

$$\frac{r}{mn} v^{\frac{s}{m}+\frac{r}{mn}} \left\{ \begin{aligned} & p_{k,\frac{s}{m}+\frac{r-1}{mn}} - p_{k,\frac{s}{m}+\frac{r}{mn}} \\ & + v(p_{k,1+\frac{s}{m}+\frac{r-1}{mn}} - p_{k,1+\frac{s}{m}+\frac{r}{mn}}) \\ & + v^2(p_{k,2+\frac{s}{m}+\frac{r-1}{mn}} - p_{k,2+\frac{s}{m}+\frac{r}{mn}}) \\ & + \cdots \\ & \vdots \end{aligned} \right\}$$

$$= \frac{r}{mn} v^{\frac{1}{mn}} \left\{ v^{\frac{s}{m}+\frac{r-1}{mn}} p_{k,\frac{s}{m}+\frac{r-1}{mn}} + v^{1+\frac{s}{m}+\frac{r-1}{mn}} p_{k,1+\frac{s}{m}+\frac{r-1}{mn}} + \cdots \right\}$$

$$- \frac{r}{mn} \left\{ v^{\frac{s}{m}+\frac{r}{mn}} p_{k,\frac{s}{m}+\frac{r}{mn}} + v^{1+\frac{s}{m}+\frac{r}{mn}} p_{k,1+\frac{s}{m}+\frac{r}{mn}} + \cdots \right\}$$

$$= \frac{r}{mn} v^{\frac{1}{mn}} \mathbf{a}_{k\left|\frac{s}{m}+\frac{r-1}{mn}\right.} - \frac{r}{mn} \mathbf{a}_{k\left|\frac{s}{m}+\frac{r}{mn}\right.} \quad \ldots \ldots \quad (25)$$

—employing the notation I have explained in the earlier part of this paper, for the value of an annuity-due, deferred for a fraction of a year.

Next, summing with regard to s, giving it the values 0, 1, 2 . . . $m-1$, we have

$$\Sigma \mathbf{a}_{k\left|\frac{s}{m}+\frac{r-1}{mn}\right.} = \mathbf{a}_{k\left|\frac{r-1}{mn}\right.} + \mathbf{a}_{k\left|\frac{1}{m}+\frac{r-1}{mn}\right.} + \mathbf{a}_{k\left|\frac{2}{m}+\frac{r-1}{mn}\right.} + \ldots + \mathbf{a}_{k\left|\frac{m-1}{m}+\frac{r-1}{mn}\right.} = m\mathbf{a}^{(m)}_{k\left|\frac{r-1}{mn}\right.},$$

and the sum of all the quantities (25) will be

$$\frac{r}{n} v^{\frac{1}{mn}} \mathbf{a}^{(m)}_{k\left|\frac{r-1}{mn}\right.} - \frac{r}{n} \mathbf{a}^{(m)}_{k\left|\frac{r}{mn}\right.}.$$

Here it will be noticed with reference to our notation that, $a^{(m)}_k$ denoting the value of an annuity of £1 payable by m instalments in each year; $\mathbf{a}^{(m)}_k$ denotes the value of a similar annuity, with the difference that the first payment is made at once instead of at the end of $\frac{1}{m}$th of a year; and $\mathbf{a}^{(m)}_{k|\tau}$ denotes the value of a similar annuity deferred τ years, the first payment of which is therefore made at the end of the time τ, which may be either integral or fractional.

We have now to find the expressions for $\mathbf{a}^{(m)}_{k\left|\frac{r-1}{mn}\right.}$ and $\mathbf{a}^{(m)}_{k\left|\frac{r}{mn}\right.}$ analogous to that found above for $\mathbf{a}_{k\left|\frac{r}{m}\right.}$, viz.—

$$a_k + \frac{m-r}{m} - \frac{r(m-r)}{2m^2}(\mu+\delta) - \frac{r(m-r)(m-2r)}{12m^3}\frac{D''_k}{D_k} \quad . \quad . \quad (26)$$

Now putting σ for $\frac{1}{m}$, we have

$$\mathbf{a}_{k\overline{|\tau}}=a_k+1-\tau-\frac{\tau(1-\tau)}{2}(\mu+\delta)-\frac{\tau(1-\tau)(1-2\tau)}{12}\frac{\mathrm{D}''_k}{\mathrm{D}_k}$$

$$\mathbf{a}_{k\overline{|\sigma+\tau}}=a_k+1-\sigma-\tau-\frac{\sigma+\tau}{2}(1-\sigma-\tau)(\mu+\delta)-\frac{\sigma+\tau}{12}(1-\sigma-\tau)(1-2\sigma-2\tau)\frac{\mathrm{D}''_k}{\mathrm{D}_k}$$

$$\mathbf{a}_{k\overline{|2\sigma+\tau}}=a_k+1-2\sigma-\tau-\frac{2\sigma+\tau}{2}(1-2\sigma-\tau)(\mu+\delta)-\frac{2\sigma+\tau}{12}(1-2\sigma-\tau)(1-4\sigma-2\tau)\frac{\mathrm{D}''_k}{\mathrm{D}_k}$$

&c. $=$ &c.

$$\mathbf{a}_{k\overline{|(m-1)\sigma+\tau}}=a_k+1-(m-1)\sigma-\tau-\frac{(m-1)\sigma+\tau}{2}(1-\overline{m-1}.\sigma-\tau)(\mu+\delta)-\frac{(m-1)\sigma+\tau}{12}(1-\overline{m-1}.\sigma-\tau)(1-2.\overline{m-1}.\sigma-2\tau)\frac{\mathrm{D}''_k}{\mathrm{D}_k}.$$

Adding these together, the sum is the annuity we have denoted by $m\mathbf{a}^{(m)}_{k\overline{|\tau}}$; and we have

$$m\mathbf{a}^{(m)}_{k\overline{|\tau}}=ma_k+m-\frac{m(m-1)}{2}\sigma-m\tau-\frac{\mu+\delta}{2}\left\{\begin{array}{l}\tau+\sigma+\tau+2\sigma+\tau+\ldots\ldots\ldots+\overline{m-1}.\sigma+\tau\\ -\tau^2-(\sigma+\tau)^2-(2\sigma+\tau)^2-\ldots-(\overline{m-1}.\sigma+\tau)^2\end{array}\right\}$$

$$-\tfrac{1}{12}\frac{\mathrm{D}''_k}{\mathrm{D}_k}\Big\{\tau+\sigma+\tau+2\sigma+\tau+\ldots\ldots\ldots+\overline{m-1}.\sigma+\tau-3[\tau^2+(\sigma+\tau)^2+(2\sigma+\tau)^2+\ldots\ldots+(\overline{m-1}.\sigma+\tau)^2]$$
$$+2[\tau^3+(\sigma+\tau)^3+(2\sigma+\tau)^3+\ldots\ldots+(\overline{m-1}.\sigma+\tau)^3]\Big\}$$

$$=ma_k+m-\frac{m(m-1)}{2}\sigma-m\tau-\frac{\mu+\delta}{2}\left\{m\tau+\frac{m(m-1)}{2}\sigma-m\tau^2-m(m-1)\sigma\tau-\frac{(m-1)m(2m-1)}{6}\sigma^2\right\}$$

$$-\tfrac{1}{12}\frac{\mathrm{D}''_k}{\mathrm{D}_k}\Big\{m\tau+\frac{m(m-1)}{2}\sigma-3\left[m\tau^2+m(m-1)\sigma\tau+\frac{(m-1)m(2m-1)}{6}\sigma^2\right]$$
$$+2\left[m\tau^3+\tfrac{3}{2}m(m-1)\sigma\tau^2+\frac{3(m-1)m(2m-1)}{6}\sigma^2\tau+\frac{m^2(m-1)^2}{4}\sigma^3\right]\Big\}$$

which, since $\sigma=\frac{1}{m}$,

$$=ma_k+m-\frac{m-1}{2}-m\tau-\frac{\mu+\delta}{2}\left\{m\tau+\frac{m-1}{2}-m\tau^2-(m-1)\tau-\frac{(m-1)(2m-1)}{6m}\right\}$$

$$-\tfrac{1}{12}\frac{D''_k}{D_k}\left\{m\tau+\frac{m-1}{2}-3\left[m\tau^2+(m-1)\tau+\frac{(m-1)(2m-1)}{6m}\right]+2\left[m\tau^3+\tfrac{3}{2}(m-1)\tau^2+\frac{(m-1)(2m-1)}{2m}\tau+\frac{(m-1)^2}{4m}\right]\right\}$$

$$=ma_k+\frac{m+1}{2}-m\tau-\frac{\mu+\delta}{2}\left\{\frac{m^2-1}{6m}+\tau-m\tau^2\right\}-\tfrac{1}{12}\frac{D''_k}{D_k}\left\{\frac{\tau}{m}-3\tau^2+2m\tau^3\right\}.$$

Dividing by m,

$$\mathbf{a}^{(m)}_{k\overline{|\tau}}=a_k+\frac{m+1}{2m}-\tau-\frac{\mu+\delta}{12m^2}(m^2-1+6m\tau-6m^2\tau^2)-\frac{1}{12m^2}\frac{D''_k}{D_k}\cdot\tau(1-m\tau)(1-2m\tau) \quad . \quad . \quad . \quad . \quad (27)$$

In order to test this result, first make $m=1$, or suppose the annuity payable yearly, then the equation reduces to

$$\mathbf{a}_{k\overline{|\tau}}=a_k+1-\tau-\frac{\mu+\delta}{2}\tau(1-\tau)-\tfrac{1}{12}\frac{D''_k}{D_k}\tau(1-\tau)(1-2\tau) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

—which agrees with (26) when we substitute $\frac{r}{m}$ for τ.

Next make $\tau=0$, then the equation becomes $\mathbf{a}^{(m)}_k=a_k+\frac{m+1}{2m}-\frac{m^2-1}{12m^2}(\mu+\delta)$.

But $\mathbf{a}^{(m)}_k=\frac{1}{m}+a^{(m)}_k$, so that this gives $a^{(m)}_k=a_k+\frac{m-1}{2m}-\frac{m^2-1}{12m^2}(\mu+\delta)$

—Mr. Woolhouse's formula for the value of an annuity payable m times a year.

Again, make $\tau=\frac{1}{2m}$ in (27); then we get

$$\mathbf{a}^{(m)}_{k\overline{|\frac{1}{2m}}}=a_k+\frac{m+1}{2m}-\frac{1}{2m}-\frac{\mu+\delta}{12m^2}\left(m^2-1+3-\frac{3}{2}\right),$$

the other term disappearing,

$$=a_k+\tfrac{1}{2}-\frac{2m^2+1}{24m^2}(\mu+\delta),$$

which, making $m=1$, agrees with (10).

Here make $m=2$, then $\mathbf{a}^{(2)}_{k\overline{|\frac{1}{4}}}=a_k+\frac{1}{2}-\frac{3}{32}(\mu+\delta)$ . . . (29)

" $m=4$, " $\mathbf{a}^{(4)}_{k\overline{|\frac{1}{8}}}=a_k+\frac{1}{2}-\frac{11}{128}(\mu+\delta)$ . . . (30)

(*To be continued.*)

CORRESPONDENCE.

"EXPECTATION OF LIFE."

To the Editor.

SIR,—Is there anywhere to be found an exact and accurate definition of this function, called by some writers "expectation of life," by others "mean duration of life"? Mr. Peter Gray, in his valuable work entitled *Tables and Formulæ for the Computation of Life Contingencies*, has taken great pains to point out the errors of some preceding writers (see chapter v., pp. 59 to 72); and while treating the subject himself in a clear and satisfactory way, has been rather severe upon the inaccuracies of others. He therefore cannot, I think, reasonably object, if I direct attention to what seems to me a serious inaccuracy in his definition. He says, "By the mean duration of life at a specified age and according to a given table of mortality, is implied, the *average* number of years that, in the case of a single life, will be enjoyed by each individual of the specified age." WILL BE *enjoyed by* EACH *individual!!!* It is of course obvious that *each* individual cannot enjoy the *average* number of years, and that the preceding definition must be amended, by substituting the "average number of years which persons of the specified age, taken one with another, enjoy according to the given table of mortality."

Several writers object wholly to the use of the phrase "expectation of life"; and I notice in particular that Professor De Morgan in this *Journal*, vol. xii. p. 33, speaks of "the average life, or expectation, as it is *wrongly* called." Now, I can understand the objection made to the phrase, that it